

# A few remarks on the Generalized Vanishing Conjecture

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## Abstract

We show that the Generalized Vanishing Conjecture

$$\forall_{m \geq 1} [\Lambda^m f^m = 0] \implies \forall_{m \gg 0} [\Lambda^m (gf^m) = 0]$$

for a fixed differential operator  $\Lambda \in k[\partial]$  follows from a special case of it, namely that the additional factor  $g$  is a power of the radical polynomial  $f$ . Next we show that in order to prove the Generalized Vanishing Conjecture (up to some bound on the degree of  $\Lambda$ ), we may assume that  $\Lambda$  is a linear combination of powers of distinct partial derivatives. At last, we show that the Generalized Vanishing Conjecture holds for products of linear forms in  $\partial$ , in particular homogeneous differential operators  $\Lambda \in k[\partial_1, \partial_2]$ .

*Key words:* Generalized Vanishing Conjecture, Jacobian Conjecture, differential operator, Weyl algebra.

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## Introduction

The Jacobian Conjecture has been the topic of many papers (see [BCW] and [vdE] and its references). Until recently, there was no framework available in which this notorious conjecture could be studied. Based on work in [dBvdE], Wenhua Zhao published several papers ([Zha1], [Zha2], [Zha3], [Zha4]) which have changed this situation dramatically.

In these papers, he introduced various conjectures which imply the Jacobian Conjecture. One of these conjectures is the so-called Generalized Vanishing Conjecture. To describe it, we fix the following notations. Let  $k[x] = k[x_1, \dots, x_n]$

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be the polynomial ring in  $n$  variables over a field  $k$ . By  $D = k[\partial_1, \dots, \partial_n]$  we denote the ring of differential operators with constant coefficients. By  $k$ -linearity of taking partial derivative, the following defines  $\Lambda f \in k[x]$  with  $\Lambda \in D$  and  $f \in k[x]$  uniquely:

$$(\Lambda_1 + \Lambda_2)f = \Lambda_1 f + \Lambda_2 f \quad (\Lambda_1 \Lambda_2)f = \Lambda_1(\Lambda_2 f) \quad \partial_i f = \frac{\partial}{\partial x_i} f,$$

where  $\Lambda_1, \Lambda_2 \in D$  and  $f \in k[x]$ .

**Generalized Vanishing Conjecture (GVC).** *Let  $\Lambda \in D$  and  $f \in k[x]$  such that*

$$\Lambda^m f^m = 0 \text{ for all } m \geq 1.$$

*Then for all  $g \in k[x]$ , we have*

$$\Lambda^m (gf^m) = 0 \text{ for all } m \gg 0.$$

It was shown in [Zha2, Th. 7.2] that for a field  $k$  of characteristic zero, a positive answer to this conjecture (in all dimensions), with  $\Lambda$  being the Laplace operator  $\Delta$  (and  $g = f$ ), implies the Jacobian Conjecture. For a field of positive characteristic  $p$ , the GVC can easily be proved, because  $\Lambda^p g = 0$  for all  $\Lambda \in k[\partial]$  with trivial constant part and all  $g \in k[x]$ .

The main results of this paper can be described as follows. First we show that the  $g$ 's in the formulation of the GVC can be replaced by powers of  $f$ . We will do that in a corollary of the following theorem.

**Theorem 1.** *Let  $\tilde{f}, g \in k[x]$  and  $m \geq d$ . Suppose that*

$$\Lambda^{m-d} \tilde{f} = 0$$

*for some  $\Lambda \in D$ . If  $\deg g \leq d$ , then*

$$\Lambda^m (g\tilde{f}) = 0$$

*as well.*

**Corollary 2.** *The GVC (for some  $\Lambda \in D$ ) is equivalent to the following statement: if  $f \in k[x]$  is such that*

$$\Lambda^m f^m = 0 \text{ for all } m \geq 1,$$

*then for each  $d \geq 1$ , we have*

$$\Lambda^m f^{m+d} = 0 \text{ for all } m \gg 0.$$

*Proof.* The statement of corollary 2 follows from the GVC (for  $\Lambda \in D$ ) by taking  $g = f^d$ , so it remains to prove the converse. For that purpose, let  $g \in k[x]$  and choose  $d \geq \deg g$ . Combining the condition  $\Lambda^m f^m = 0$  for all  $m \geq 1$  of the GVC (for  $\Lambda \in D$ ) and the statement of corollary 2, we get  $\Lambda^m f^{m+d} = 0$  for all  $m \gg 0$ , which is equivalent to  $\Lambda^{m-d} f^m = 0$  for all  $m \gg 0$ . By taking  $\tilde{f} = f^m$  in theorem 1, we subsequently obtain  $\Lambda^m(g f^m) = 0$  for all  $m \gg 0$ .  $\square$

In the proof of [vdEZ, Th. 1.5], corollary 2 is proved for  $\Lambda = \Delta$ , the Laplace operator. The claim of [vdEZ, Th. 1.5] is that one can even take  $d = 1$  in corollary 2 when  $\Lambda = \Delta$ , which subsequently follows from (3)  $\Rightarrow$  (2) of [Zha2, Th. 6.2]. Hence we can take  $g = f$  in the GVC when we restrict ourselves to  $\Lambda = \Delta$ .

If  $k$  has characteristic zero, then by [Cou, §1.1], we can also formulate theorem 1 and corollary 2 in terms of the  $n$ -th Weyl algebra  $A_n(k)$  over  $k$ .  $A_n(k)$  is the algebra of skew polynomials in  $x$  and  $\partial$  over  $k$ , with the following commutator relations, where  $\delta$  is Kronecker's delta:

$$x_i x_j - x_j x_i = \partial_i \partial_j - \partial_j \partial_i = 0 \quad \partial_i x_j - x_j \partial_i = \delta_{ij} \text{ for all } i, j.$$

We get the Weyl algebra formulation of theorem 1 and corollary 2 as follows. In each of the equalities of the form  $E = 0$  in them, we interpret the left hand side  $E$  as an element of  $A_n(k)$ , and replace ' $E = 0$ ' by ' $E$  is contained in the left ideal of  $A_n(k)$  generated by  $\partial_1, \partial_2, \dots, \partial_n$ '. See [Cou, §5.1] for the justification of this reformulation, especially [Cou, Prop. 5.1.2].

If  $k$  has positive characteristic, then the Weyl algebra formulations of theorem 1 and corollary 2 are not just reformulations of theorem 1 and corollary 2, but really different claims, see [Cou, §2.3]. This is however not essential for the proof of corollary 2, and neither will be essential for the proof of theorem 1 in the next section, so we actually have two theorems 1 and corollaries 2.

Next we show that it suffices to investigate the GVC for a special class of operators. More precisely we show the following.

**Theorem 3.** *It suffices to investigate the GVC in all dimensions for operators of the form*

$$c_1 \partial_1^{d_1} + c_2 \partial_2^{d_2} + \dots + c_n \partial_n^{d_n},$$

where the  $d_i$  are positive integers and  $c_i \in k$ . Furthermore, we may take  $c_i = 1$  or  $c_i^2 = 1$  for all  $i$  when  $k = \mathbb{C}$  or  $k = \mathbb{R}$  respectively.

Finally we prove the GVC for the following class of operators.

**Theorem 4.** *Let  $\Lambda \in D$  be a product of linear forms (in the  $\partial_i$ ). Then the GVC holds for  $\Lambda$ .*

As a consequence, we can deduce the following.

**Corollary 5.** *In dimension two, the GVC holds for any homogeneous operator.*

*Proof.* Assume without loss of generality that  $k$  is algebraically closed. Then any homogeneous polynomial in two variables is a product of linear factors. Hence the result follows from theorem 4.  $\square$

## The proofs

Fix  $\tilde{f} \in k[x]$ . For  $\Lambda \in D$  and  $f, g \in k[x]$ , define

$$[\Lambda, g]f := \Lambda(gf) - g(\Lambda f).$$

Notice that by the product rule of differentiation,

$$[\partial_i, g]f = \partial_i(gf) - g(\partial_i f) = g_{x_i}f \quad (1)$$

for all  $f, g \in k[x]$ , where  $g_{x_i}$  is the polynomial  $\frac{\partial}{\partial x_i}g$ . More generally, write  $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}$ . If  $\alpha_i = 0$  for all  $i$ , then trivially  $[\partial^\alpha, g]\tilde{f} = 0$  for all  $g \in k[x]$ . Hence assume that  $\alpha_i \neq 0$  for some  $i$  and define  $\partial^{\hat{\alpha}}$  by  $\partial^{\hat{\alpha}} \partial_i = \partial^\alpha$ . Using (1) with  $f = \partial^{\hat{\alpha}} \tilde{f}$ , we obtain

$$\begin{aligned} [\partial^\alpha, g]\tilde{f} &= \partial^\alpha(g\tilde{f}) - g(\partial^\alpha \tilde{f}) \\ &= \partial_i(\partial^{\hat{\alpha}}(g\tilde{f}) - g(\partial^{\hat{\alpha}} \tilde{f})) + \partial_i(g(\partial^{\hat{\alpha}} \tilde{f})) - g(\partial_i(\partial^{\hat{\alpha}} \tilde{f})) \\ &= \partial_i([\partial^{\hat{\alpha}}, g]\tilde{f}) + [\partial_i, g](\partial^{\hat{\alpha}} \tilde{f}) \\ &= \partial_i([\partial^{\hat{\alpha}}, g]\tilde{f}) - g_{x_i}(\partial^{\hat{\alpha}} \tilde{f}) \\ &= \partial_i([\partial^{\hat{\alpha}}, g]\tilde{f}) + \partial^{\hat{\alpha}}(g_{x_i} \tilde{f}) - [\partial^{\hat{\alpha}}, g_{x_i}]\tilde{f} \end{aligned}$$

for all  $g \in k[x]$ . By induction on  $\sum_{i=1}^n \alpha_i$ , it follows that  $[\partial^\alpha, g]\tilde{f}$  can be expressed as a  $k$ -linear combination of polynomials of the form  $\Lambda^*(g^* \tilde{f})$ , where  $\Lambda^* \in D$  and  $g^* \in k[x]$  is a polynomial of degree less than  $\deg g$ .

Now fix  $g \in k[x]$  as well and set  $d := \deg g$ . If we define  $D_{[x]}^{(r)} \tilde{f}$  as the  $k$ -space of polynomials  $\Lambda^*(g^* \tilde{f})$ , with  $\Lambda^* \in D$  and  $g^* \in k[x]$  of degree  $\leq r$ , then  $[\partial^\alpha, g]\tilde{f} \in D_{[x]}^{(d-1)} \tilde{f}$ . Writing an arbitrary operator  $\Lambda \in D$  as a  $k$ -linear combination of monomials  $\partial^\alpha$ , we deduce that

$$[\Lambda, g]\tilde{f} \in D_{[x]}^{(d-1)} \tilde{f} \text{ for all } \Lambda \in D. \quad (2)$$

*Proof of theorem 1.* Assume  $\deg g \leq d$ . Recall that we have  $\Lambda^{m-d} \tilde{f} = 0$  and must show that the left hand side  $\Lambda^m(g\tilde{f})$  of (3) below is zero. If  $m = d$ , then  $\tilde{f} = 0$ . If  $d = 0$ , then  $g \in k$ . Hence the cases  $m = d$  and  $d = 0$  are trivial. So assume that  $m > d > 0$ . Notice that  $\Lambda(g\tilde{f}) = g(\Lambda \tilde{f}) + [\Lambda, g]\tilde{f}$ , whence

$$\Lambda^m(g\tilde{f}) = \Lambda^{m-1}(g(\Lambda \tilde{f})) + \Lambda^{m-1}([\Lambda, g]\tilde{f}). \quad (3)$$

Since  $\Lambda^{(m-1)-d}(\Lambda \tilde{f}) = \Lambda^{m-d}(\tilde{f}) = 0$ , it follows by induction on  $m$  that the first term  $\Lambda^{m-1}(g(\Lambda \tilde{f})) = 0$  on the right hand side of (3) vanishes.

Hence it suffices to show that the second term  $\Lambda^{m-1}([\Lambda, g]\tilde{f})$  on the right hand side of (3) vanishes as well. For that purpose, we use (2) to write  $[\Lambda, g]\tilde{f}$  as a sum of polynomials of the form  $\Lambda^*(g^* \tilde{f})$ , where  $\Lambda^* \in D$  and  $g^* \in k[x]$  such

that  $\deg g^* \leq d-1$  for each such term. Thus the second term  $\Lambda^{m-1}([\Lambda, g]\tilde{f})$  on the right hand side of (3) is a sum of polynomials of the form

$$\Lambda^{m-1}\Lambda^*(g^*\tilde{f}) = \Lambda^*(\Lambda^{m-1}(g^*\tilde{f})), \quad (4)$$

with  $\Lambda^*$  and  $g^*$  as above. Since  $\Lambda^{(m-1)-(d-1)}(\tilde{f}) = \Lambda^{m-d}(\tilde{f}) = 0$ , it follows by induction on  $d$  that the second factor on the right hand side of (4) vanishes. So both sides of (3) are zero.  $\square$

Just as theorem 1 itself, the above proof can be reformulated in terms of the  $n$ -th Weyl algebra  $A_n(k)$  over  $k$  (resulting in the proof of a different theorem when  $k$  has positive characteristic). The Weyl algebra formulation of (1) is [Cou, Exrc. 1.4.1] and the definition of  $D_{[x]}^{(r)}\tilde{f}$  corresponds to a filtration which is isomorphic to the order filtration in [Cou, §7.2], see also [Cou, §3.2]. The actual isomorphism is generated by

$$x_i \mapsto \partial_i \quad \partial_i \mapsto -x_i \quad \text{for all } i.$$

By way of the same isomorphism, it follows from [Cou, Prop. 1.2.1] that every element of  $A_n(k)$  is a linear combination of elements of the form  $\Lambda^*g^*$ , where  $\Lambda^* \in D$  and  $g^* \in k[x]$ . This fact has some connection with (2).

*Proof of theorem 3.* Let  $\Lambda \in D$  be a non-zero operator and  $f \in k[x]$  such that  $\Lambda^m f^m = 0$  for all  $m \geq 1$ . We must show that  $\Lambda^m(gf^m) = 0$  in case the GVC holds for  $k$ -linear combinations of powers of distinct partial derivatives.

- i) Since each monomial of degree  $d$  appearing in  $\Lambda$  can be written as a  $k$ -linear combination of powers of the form  $l^d$ , where  $l$  is a  $k$ -linear combination of  $\partial_1, \partial_2, \dots, \partial_n$ , we can write  $\Lambda$  as a  $k$ -linear combination of such powers, i.e.

$$\Lambda = c_1 l_1^{d_1} + c_2 l_2^{d_2} + \dots + c_N l_N^{d_N} \quad (5)$$

for some  $c_i \in k^*$ ,  $d_i \in \mathbb{N}$ , where  $l_i = a_{i1}\partial_1 + \dots + a_{in}\partial_n$ , for some  $a_{ij} \in k$ . If  $k = \mathbb{C}$  or  $k = \mathbb{R}$ , then we may assume that  $c_1 = c_2 = \dots = c_n = 1$  or  $c_1^2 = c_2^2 = \dots = c_n^2 = 1$  respectively.

- ii) Now we introduce  $N$  new variables  $y_1, y_2, \dots, y_N$  and consider the operator

$$\Lambda^* := (\partial_{y_1} + l_1)^{d_1} + (\partial_{y_2} + l_2)^{d_2} + \dots + (\partial_{y_N} + l_N)^{d_N} \quad (6)$$

on the polynomial ring  $k[x_1, \dots, x_n, y_1, \dots, y_N]$ . Making the linear coordinate change defined by

$$x'_i := x_i - (a_{1i}y_1 + \dots + a_{Ni}y_N) \quad y'_j := y_j$$

for all  $i \leq n$  and for all  $j \leq N$ , we obtain  $\partial_{y_j} + l_j = \partial_{y'_j}$ , because

$$(\partial_{y_j} + l_j)x'_i = -a_{ji} + a_{ji} = 0 \quad (\partial_{y_j} + l_j)y'_t = \delta_{jt}$$

for all  $i \leq n$  and all  $j, t \leq N$ . Hence on these new coordinates, the operator  $\Lambda^*$  is of the form as described in the statement of the theorem. It follows that the GVC holds for  $\Lambda^*$ .

- iii) Since  $f \in k[x]$  satisfies  $\Lambda^m f^m = 0$  for all  $m \geq 1$ , it follows from (5) and (6) that also  $(\Lambda^*)^m f^m = 0$  for all  $m \geq 1$ . As observed in ii), the GVC holds for  $\Lambda^*$ , thus we obtain that for every  $g \in k[x]$  also  $(\Lambda^*)^m (gf^m) = 0$  for all large  $m$ . But again by (5) and (6), it follows that  $\Lambda^m (gf^m) = 0$  for all large  $m$ , which concludes the proof.  $\square$

Finally, we give the proof of theorem 4, in which we re-use some techniques given in the proof of theorem 3.

*Proof of theorem 4.* Let  $\Lambda = l_1 \cdot l_2 \cdot \dots \cdot l_N$  for some non-zero linear forms  $l_i$  in  $\partial_1, \partial_2, \dots, \partial_n$ . As above, introduce  $N$  new variables  $y_1, y_2, \dots, y_N$  and consider the operator

$$\Lambda^* := (\partial_{y_1} + l_1) \cdot (\partial_{y_2} + l_2) \cdot \dots \cdot (\partial_{y_N} + l_N) \quad (7)$$

Making the linear coordinate change

$$x'_i := x_i - (a_{1i}y_1 + \dots + a_{Ni}y_N) \quad y'_j := y_j$$

for all  $i \leq n$  and for all  $j \leq N$  again, we obtain that on these new coordinates, the operator  $\Lambda^*$  is of the form

$$\partial_{y'_1} \cdot \partial_{y'_2} \cdot \dots \cdot \partial_{y'_N}.$$

By [vdEWZ, Cor. 3.5.3], it follows that the GVC holds for this operator. Hence the GVC holds for  $\Lambda^*$ . So just as in iii) above, we deduce that the GVC holds for  $\Lambda$ .  $\square$

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